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# Stochastic uniform observability of linear differential equations with multiplicative noise

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**Abstract**

The aim of this paper is to give a deterministic characterization of the uniform observability property of linear differential equations with multiplicative white noise in infinite dimensions. We also investigate the properties of a class of perturbed evolution operators and we used these properties to give a new representation of the covariance operators associated to the mild solutions of the investigated stochastic differential equations. The obtained results play an important role in obtaining necessary and sufficient conditions for the stochastic uniform observability property.


*Keywords:* Stochastic differential equations; Perturbed evolution operators; Covariance operators; Stochastic uniform observability; Hilbert–Schmidt spaces

## 1. Introduction

Since 1960, when Kalman published his famous paper [6] where he introduced the concepts of controllability, observability, uniform controllability and uniform observability for deterministic linear control systems, these properties have been extensively studied both for deterministic and stochastic cases.

The main purpose of this paper is to give a deterministic characterization of the stochastic uniform observability property of a class of linear stochastic differential equations with multiplicative noise defined on infinite dimensional Hilbert spaces. We adopted here the definition of stochastic uniform observability introduced by Morozan in [13]. This definition was inspired by the concept of stochastic uniform observability given by Zabczyk in [20] for stochastic discrete time linear systems with constants coefficients. There are some papers, which proved that under stochastic uniform observability and stabilizability conditions there exist some global solutions of the differential Riccati equations arising in stochastic control (see [2,13–16,18]). Similar results were obtained before [9,10] under detectability and stabilizability conditions. Since the concept of stochastic observability considered in this paper does not imply detectability (see [2,16]) as it happens in the deterministic case, the results obtained under stochastic uniform observ-

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ability conditions are different to those obtained under detectability hypothesis. Then, we can ask a question: Why to choose stochastic uniform observability instead of detectability? The answer could be related to the difficulty of verification of each of these two properties. Therefore the main result of this paper (Theorem 19), which gives a deterministic characterization of the stochastic uniform observability property by using Lyapunov functions, could help us to decide if the stochastic uniform observability property is easy to verify or not. Corollary 22 particularizes the result of Theorem 19 to the time invariant case and also proves that our result extends to the infinite dimensional case (Theorem 4.3 in [2]). In order to prove Theorem 19 we need to establish a series of results concerning a class of perturbed evolution operators and to give a new representation of the covariance operators associated to the mild solutions of the investigated stochastic differential equations. As we will show later the covariance operator could be expressed by using a perturbed evolution operator defined on certain Hilbert–Schmidt space. This new representation of the covariance operator is different to those obtained in [17], where was obtained only a formula for the nuclear norm of the covariance operator.

## 2. Preliminaries and statement of the problem

Let $H, V$ be separable real Hilbert spaces. $L(H,V)$ is the Banach space of all bounded linear operators from $H$ into $V$ (if $H=V$ we put $L(H,V)=L(H)$). We denote by $I_H$ or simply $I$ the identity operator on $H$. We write $\langle .,. \rangle$ for the inner product and $\|.\|$ for norms of elements and operators. We will say that $S \in L(H)$ is nonnegative (and we will write $S \geqslant 0$) if $S$ is self-adjoint and $\langle Sx, x\rangle \geqslant 0$, $x \in H$. We denote by $L^+(H)$ the subset of $L(H)$ of all nonnegative operators. The cone $L^+(H)$ introduce on $L(H)$ the following order: $S_1 \geqslant S_2$ iff $S_1 - S_2 \in L^+(H)$. We will say that $S \in L^+(H)$ is positive (and we will write $S > 0$) if there exists $\gamma > 0$ such that $S \geqslant \gamma I$.

Let $P \in L^+(H)$ and $S \in L(H)$. We denote by $P^{1/2}$ the square root of $P$, by $S^*$ the adjoint operator of $S$ and by $|S|$ the operator $(S^*S)^{1/2}$. We put $\|S\|_1 = \operatorname{Tr}|S| \leqslant \infty$ and we also denote by $C_1(H)$ the set $\{S \in L(H)/\|S\|_1 < \infty\}$ (the trace class of operators). If $S \in C_1(H)$ we say that $S$ is nuclear.

The definition of nuclear operators introduced above is equivalent with that given in [3] and [5].

If $\|S\|_2 = (\operatorname{Tr} S^*S)^{1/2}$ we can introduce the Hilbert–Schmidt class of operators, namely $C_2(H) = \{S \in L(H)/\|S\|_2 < \infty\}$ (see [11]).

It is known that $C_2(H)$ is a Hilbert space with the inner product $\langle S, T\rangle^2 = \operatorname{Tr} T^*S$, $T, S \in C_2(H)$ (see [11]).

We denote by $\mathscr{H}_2$ the subspace of $C_2(H)$ of all self-adjoint operators. Since $\mathscr{H}_2$ is closed in $C_2(H)$ with respect to $\|.\|_2$ we deduce that it is a Hilbert space, too. For any $S \in C_1(H)$ we have (see [5]):

$$\|S\| \leqslant \|S\|_2 \leqslant \|S\|_1.$$

We recall the following property of sequences in $C_1(H)$, respectively in $C_2(H)$.

**Lemma 1.** *(See [3,4].) If the sequence $\{C_n\}_{n\in\mathbf{N}} \subset L(H)$ is strongly convergent to $C \in L(H)$ and $S \in C_1(H)$ (respectively $S \in C_2(H)$) then $C_nS$ converges to $CS$ in $\|.\|_1$ (respectively in $\|.\|_2$).*

Let $J \subset \mathbf{R}_+ = [0,\infty)$ be an interval. If $E$ is a Banach space we denote by $C(J,E)$ the space of all mappings $G: J \to E$ that are continuous. We also denote by $C_s(J, L(H))$ the space of all strongly continuous mappings $G: J \to L(H)$.

Let $T > 0$ and let us denote $\Delta(T) = \{(t,s),\ 0 \leqslant s \leqslant t \leqslant T\}$ and $\Delta_p(T) = \{(s_p,\dots,s_2,s_1),\ 0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_p \leqslant T\}$, $p \in \mathbf{N}$, $p > 3$. If $T = \infty$, we set $\Delta = \Delta(\infty)$. Then $C_s(\Delta(T), L(H))$ (respectively $C_s(\Delta_p(T), L(H))$) will be the set of all strongly continuous mappings $G: \Delta(T) \to L(H)$ (respectively $G: \Delta_p(T) \to L(H)$).

It is not difficult to see that $C_s(\Delta(T), L(H))$, respectively $C_s(\Delta_p(T), L(H))$ are Banach spaces when endowed them with the usual operations and with the norm

$$|||G|||_T = \sup_{(t,s)\in\Delta(T)} \|G(t,s)\|,$$

respectively

$$|||G|||_T = \sup_{(s_p,s_{p-1},\dots,s_1)\in\Delta_p(T)} \|G(s_p,s_{p-1},\dots,s_1)\|.$$

Throughout this paper we will assume, if we will not specify otherwise, that the following two hypotheses hold:

**P1:** (a) $A(t),\ t \in [0, \infty)$ is a closed linear operator on $H$ with constant domain $D$ dense in $H$;
(b) there exist $M > 0$, $\eta \in (\frac{1}{2}\pi, \pi)$ and $\delta \in (-\infty, 0)$ such that $S_{\delta,\eta} = \{\lambda \in C;\ |\arg(\lambda - \delta)| < \eta\} \subset \rho(A(t))$, for all $t \geqslant 0$ and for all $\lambda \in S_{\delta,\eta}$,

$$\|R(\lambda, A(t))\| \leqslant \frac{M}{|\lambda - \delta|};$$

(c) there exist numbers $\alpha \in (0, 1)$ and $\widetilde{N} > 0$ such that

$$\|A(t)A^{-1}(s) - I\| \leqslant \widetilde{N}|t - s|^{\alpha}, \quad t \geqslant s \geqslant 0,$$

where we denote by $\rho(A)$, $R(\lambda, A)$ the resolvent set of $A$ and respectively the resolvent of $A$.

If a family $A(t),\ t \in [0, \infty)$ satisfies the hypothesis (P1) we will say that P1$(A)$ holds. For example, in some of the next sections we will assume that $A^*(t),\ t \in [0, \infty)$ also satisfies (P1) and we will write that P1$(A^*)$ holds.

**P2:** $B \in C_s([0, T], L(H))$, $G_i \in C_s([0, T], L(H))$, $i \in \{1, \ldots, m\}$, $m \in \mathbf{N}^*$.

Another hypothesis which will be useful in the subsequent considerations is:

**P3:** $B^* \in C_s([0, T], L(H))$, $G_i^* \in C_s([0, T], L(H))$, $i \in \{1, \ldots, m\}$, $m \in \mathbf{N}^*$.

Now, let us introduce the following definition:

**Definition 2.** (See Definition 5.3 in [7].) A family $\{V(t, s)\}_{(t,s)\in\Delta(T)} \subset L(H)$, is an evolution operator (system) iff

1. $V \in C_s(\Delta(T), L(H))$ and
2. $V(s, s) = I$, $V(t, s)V(s, r) = V(t, r)$ for all $0 \leqslant r \leqslant s \leqslant t \leqslant T$ (semigroup property).

It is known [7,12] that if (P1) holds, then the family $A(t)$, $t \geqslant 0$ (often we shall denote the family $A(t)$ by $A$ to avoid complicated notations) generates an evolution operator $U = U(t, s)$, which also has the following properties:

1. $U(t, s)(H) \subset D$ for all $(t, s) \in \Delta$;
2. $\frac{\partial U(t,s)x}{\partial t} = A(t)U(t, s)x$ for every $x \in H$ and $(t, s) \in \Delta$;
3. for any $x \in D$, $U(t, s)x$ is differentiable with respect to $s$ on $0 \leqslant s \leqslant t$ and

$$\frac{\partial U(t, s)x}{\partial s} = -U(t, s)A(s)x.$$

(See [7] for other properties of evolution operators.)

By (P1)(b) it follows that for all $n \in \mathbf{N}$ we have $n \in \rho(A(t))$ and consequently the operators $A_n(t) = n^2R(n, A(t)) - nI$ (called the Yosida approximations of $A(t)$) are well defined. It is known, that $A_n \in C([0, T], L(H))$ (see Lemma 3 in [17]) and it generates an evolution operator $\{U_n(t, s)\}_{(t,s)\in\Delta} \subset L(H)$ (Theorem 5.2 in [7]). Moreover, for each $x \in H$ one has

$$\lim_{n\to\infty} U_n(t, s)x = U(t, s)x \tag{1}$$

uniformly on any bounded subset of $\Delta$ (see also [12]).

Let $T > 0$. We introduce the following integral equations

$$U_B(t, s)x = U(t, s)x + \int_s^t U(t, r)B(r)U_B(r, s)x\,dr, \quad x \in H,\ (t, s) \in \Delta(T), \tag{2}$$

$$U_{B,n}(t,s)x = U_n(t,s)x + \int_s^t U_n(t,r)B(r)U_{B,n}(r,s)x\,dr, \quad x \in H,\ (t,s) \in \Delta(T). \tag{3}$$

It is known (see Theorem 9.2, Corollary 9.3 in [1] and Lemma 4.5 in [7], respectively Proposition 5 in [17]) that if (P1), (P2) hold then (2), respectively (3) have unique solution $U_B(.,.)$, respectively $U_{B,n}(.,.)$ on $\Delta(T)$ in the class of strongly continuous operators in $H$. Moreover, these solutions are evolution operators on $\Delta(T)$ and we call it the perturbed evolution operators corresponding to the perturbation $B$ (defined by Eqs. (2), (3)).

Our problems are the following:

(1) To establish if the following statement is true:

$$\text{for each } x \in H, \quad \lim_{n\to\infty} U_{B,n}(t,s)x = U_B(t,s)x \quad \text{uniformly on } \Delta(T).$$

(2) To provide a representation of the covariance operator associated to the solution of the stochastic differential equation (10) by using the solution of an associated Lyapunov equation defined on a Hilbert–Schmidt space (see Theorem 19). We note that this result is better than the similar one proved in [17], where we only obtain a relation between the nuclear norm of the covariance operator and the solution of an associated Lyapunov equation.

(3) To give deterministic characterizations of the stochastic uniform observability property of the stochastic equation (10).

## 3. Perturbed evolution operators

In this section we will give a positive answer to the first problem formulated in the last section by proving that the "approximating" perturbed evolution operator $U_{B,n}$ converges strongly to the evolution operator $U_B$ uniformly on $\Delta(T)$.

In order to prove the above result, we need the following lemma, obtained by a slightly modification of Lemma 4.2 in [8]. The proof is not very long and we will present it for the reader convenience.

**Lemma 3.** *If $(P_n)_{n\in\mathbf{N}^*}$, $(Q_n)_{n\in\mathbf{N}^*}$ are two sequences in $C_s(\Delta(T), L(H))$, such that for every $x \in H$, $P_n(t,s)x \to_{n\to\infty} P(t,s)x$, $Q_n(t,s)x \to_{n\to\infty} Q(t,s)x$ uniformly with respect to $(t,s) \in \Delta(T)$, then $P_n(t,s)Q_n(t,s)x \to_{n\to\infty} P(t,s)Q(t,s)x$ uniformly with respect to $(t,s) \in \Delta(T)$. (Obviously $P, Q, PQ \in C_s(\Delta(T), L(H))$.)*

**Proof.** Using the uniform boundedness principle and the hypothesis we see that there exists $M > 0$ such that $\|P_n(t,s)\|, \|Q_n(t,s)\| < M$ for all $n \in \mathbf{N}$, $(t,s) \in \Delta(T)$. For any $x \in H$ we have

$$\big\|\big[P_n(t,s)Q_n(t,s) - P(t,s)Q(t,s)\big]x\big\| \leqslant \big\|\big[P_n(t,s) - P(t,s)\big]Q(t,s)x\big\| + M\big\|Q_n(t,s)x - Q(t,s)x\big\|.$$

Since $Q_n(t,s)x \to_{n\to\infty} Q(t,s)x$ uniformly with respect to $(t,s) \in \Delta(T)$, we only have to prove that the first member of the right side of the above inequality converges to 0, uniformly with respect to $(t,s) \in \Delta(T)$.

Let $\{e_1, e_2, \ldots, e_n, \ldots\}$ be an orthonormal basis of $H$. Then for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbf{N}$, such that

$$\left\|Q(t,s)x - \sum_{i=1}^{N_\varepsilon}\big\langle Q(t,s)x, e_i\big\rangle e_i\right\| < \varepsilon.$$

Hence

$$\big\|\big[P_n(t,s) - P(t,s)\big]Q(t,s)x\big\| \leqslant \sum_{i=1}^{N_\varepsilon}\big|\big\langle Q(t,s)x, e_i\big\rangle\big|\big\|P_n(t,s)e_i - P(t,s)e_i\big\| + 2M\varepsilon.$$

Denoting $q_{i,x} = \sup_{(t,s)\in\Delta(T)} |\langle Q(t,s)x, e_i\rangle|$ we see that

$$\big\|\big[P_n(t,s) - P(t,s)\big]Q(t,s)x\big\| \leqslant \sum_{i=1}^{N_\varepsilon} q_{i,x}\big\|P_n(t,s)e_i - P(t,s)e_i\big\| + 2M\varepsilon.$$

Now, we use the hypothesis to deduce that for all $x \in H$,

$$\left\| \left[ P_n(t,s) - P(t,s) \right] Q(t,s)x \right\| \underset{n\to\infty}{\to} 0$$

uniformly with respect to $(t,s) \in \Delta(T)$. The proof is complete. □

**Remark 4.** The statement of the above lemma remains true if we replace $\Delta(T)$, with $\Delta_p(T)$, $p \in \mathbf{N}$, $p > 3$. We preferred to prove it for $\Delta(T)$ instead of $\Delta_p(T)$ to avoid complicated notations.

**Theorem 5.** *If $U_B(.,.)$, respectively $U_{B,n}(.,.)$ are the perturbed evolution operators corresponding to the perturbation $B \in C_s([0,T], L(H))$ and defined by Eqs.* (2), (3) *then*

(1) *$U_{B,n}(t,s)$ is the unique strong* (*classical*) *solution of the equation*

$$\begin{aligned} &\frac{\partial x_n(t)}{\partial t} = \left[ A_n(t) + B(t) \right] x_n(t), \quad t \in (s,T], \\ &x_n(s) = x, \quad x \in H; \end{aligned} \tag{4}$$

(2) *for each $x \in H$, $\lim_{n\to\infty} U_{B,n}(t,s)x = U_B(t,s)x$ uniformly on $\Delta(T)$.*

**Proof.** We first note that $A_n + B \in C_s([0,T], L(H))$ by Lemma 3 in [17]. Then the statement (1) is a consequence of Proposition 5 in [17]. We only have to prove the statement (2).

*Step* I. We will show that there exists $T_0 \in (0,T]$ such that the statement (2) is true if we replace $\Delta(T)$ with $\Delta(T_0)$. Let us introduce the functions $\phi, \phi_n : C_s(\Delta(T), L(H)) \to C_s(\Delta(T), L(H))$,

$$\begin{aligned} &\phi(G)(t,s)x = U(t,s)x + \int_s^t U(t,r)B(r)G(r,s)x\,dr, \\ &\phi_n(G)(t,s)x = U_n(t,s)x + \int_s^t U_n(t,r)B(r)G(r,s)x\,dr. \end{aligned} \tag{5}$$

(a) The functions $\phi$, $\phi_n$ are well defined on $C_s(\Delta(T), L(H))$. Indeed, we deduce by the hypotheses that if $G \in C_s(\Delta(T), L(H))$ then the mapping $F : \Delta_3(T) \to L(H)$, $F(t,r,s) = U(t,r)B(r)G(r,s)$ is strongly continuous.

Thus $(t,s) \to \int_s^t U(t,r)B(r)G(r,s)x\,dr$ is continuous for any $x \in H$ and consequently $\phi(G) \in C_s(\Delta(T), L(H))$. Hence $\phi$ is well defined. The proof for $\phi_n$ is similar and is omitted.

(b) Now we will prove that there exists $T_0 \in (0,T]$ such that $\phi$, $\phi_n$ are contractions on $C_s(\Delta(T_0), L(H))$. Let us denote $M_{U,B}^T = \sup_{(t,s)\in\Delta(T)} \|U(t,s)B(s)\|$. For any $t \in (0,T]$, $G_1, G_2 \in C_s(\Delta(T), L(H))$ we have

$$\left\| \phi(G_1) - \phi(G_2) \right\|_t \leqslant t M_{U,B}^T \|G_1 - G_2\|_t.$$

Since there exists $t = T_0 > 0$ such that $T_0 M_{U,B}^T < 1$ it follows that $\phi$ is a contraction on $C_s(\Delta(T_0), L(H))$.

Analogously, we will prove that there exists $\widetilde{T}_0 \in (0,T]$ such that $\phi_{n,k}$ is a contraction on $C_s(\Delta(\widetilde{T}_0), L(H))$. Using (1) and the uniform boundedness theorem we deduce that there exists $M_U^T > 0$ such as $\|U_n(t,r) - U(t,r)\| \leqslant M_U^T$ for $(t,r) \in \Delta(T)$ and $n \in \mathbf{N}$. Thus there exists $\widetilde{M}_{U,B}^T > 0$ such that $\sup_{(t,s)\in\Delta(T)} \|U_n(t,s)B(s)\| \leqslant \widetilde{M}_{U,B}^T$ for all $n \in \mathbf{N}$. We choose $\widetilde{T}_0$ such that $\widetilde{T}_0 \widetilde{M}_{U,B}^T < 1$ and reasoning as above we deduce that $\phi_n$ is a contraction on $C_s(\Delta(\widetilde{T}_0), L(H))$.

We may assume (without the loss of the generality) that $\widetilde{T}_0 = T_0$.

(c) By the principle of contractions it follows that the equations $\phi(G) = G$, respectively $\phi_n(G) = G$, $n \in \mathbf{N}$, have unique solutions $U_B$, respectively $U_{B,n}$ in $C_s(\Delta(T_0), L(H))$.

Let $G_0(t,s) = U(t,s)$, $(t,s) \in \Delta(T_0), \ldots, G_{m+1} = \phi(G_m)$, $m = 0,1,\ldots,$ and $H_{n,0}(t,s) = U(t,s)$, $(t,s) \in \Delta(T_0), \ldots, H_{n,m+1} = \phi(H_{n,m})$, $m = 0,1,\ldots,$ $n \in \mathbf{N}$, be the sequences, which converges to $U_B$, respectively to $U_{B,n}$ in $C_s(\Delta(T_0), L(H))$.

(d) We will show that for each $x \in H$, $\lim_{n\to\infty} U_{B,n}(t,s)x = U_B(t,s)x$ uniformly on $\Delta(T_0)$. First, we will prove by induction on $m \in \mathbf{N}$ that $H_{n,m}(t,s)x \to_{n\to\infty} G_m(t,s)x$ uniformly on $\Delta(T_0)$. For $m = 0$ the statement is obviously true. We assume that the statement is true for $m \in \mathbf{N}$ and we will prove that it is also true for $m+1$. We have

$$H_{n,m+1}(t,s)x - G_{m+1}(t,s)x = \int_s^t U_n(t,r)B(r)H_{n,m}(r,s)x - U(t,r)B(r)G_m(r,s)x\,dr.$$

Using (1), the induction assumption, Lemma 3 and Remark 4 we see that

$$\lim_{n\to\infty} U_n(t,r)B(r)H_{n,m}(r,s)x = U(t,r)B(r)G_m(r,s)x$$

uniformly with respect to $(t,r,s) \in \Delta_3(T_0)$. Hence we apply the Dominated Convergence Theorem of Lebesgue to deduce that $H_{n,m+1}(t,s)x - G_{m+1}(t,s)x \to_{n\to\infty} 0$ uniformly on $\Delta(T_0)$. The proof of the induction is complete.

With the above notations we set $q = \max\{T_0 M_{U,B}^T, T_0 \widetilde{M}_{U,B}^T\} < 1$. For any $m,n \in \mathbf{N}$, $(t,s) \in \Delta(T_0)$ and $x \in H$

$$\begin{aligned}
&\bigl\| U_{B,n}(t,s)x - U_B(t,s)x \bigr\| \\
&\quad \leqslant \bigl( |||H_{n,m} - U_{B,n}|||_{T_0} + |||G_m - U_B|||_{T_0} \bigr)\|x\| + \bigl\| H_{n,m}(t,s)x - G_m(t,s)x \bigr\| \\
&\quad \leqslant q\bigl( \bigl|\bigl|\bigl| \phi_n(H_{n,m-1} - U_{B,n}) \bigr|\bigr|\bigr|_{T_0} + \bigl|\bigl|\bigl| \phi(G_{m-1} - U_B) \bigr|\bigr|\bigr|_{T_0} \bigr)\|x\| + \bigl\| H_{n,m}(t,s)x - G_m(t,s)x \bigr\|.
\end{aligned}$$

Using the properties of the contractions $\phi$ and $\phi_n$ and the induction we get

$$\begin{aligned}
\bigl\| U_{B,n}(t,s)x - U_B(t,s)x \bigr\| &\leqslant q^m \bigl( \bigl|\bigl|\bigl| U(t,s) - U_{B,n} \bigr|\bigr|\bigr|_{T_0} + \bigl|\bigl|\bigl| U(t,s) - U_B \bigr|\bigr|\bigr|_{T_0} \bigr)\|x\| \\
&\quad + \bigl\| H_{n,m}(t,s)x - G_m(t,s)x \bigr\|.
\end{aligned}$$

We recall that there exists $M > 0$ such that $\|U_n(t,s)\| < M$ for all $(t,s) \in \Delta(T_0)$. Then we apply Gronwall's inequality in (3) and we deduce that the sequence $|||U_{B,n}|||_{T_0}$ is bounded on $\mathbf{N}$, that is there exists $h > 0$ such that $|||U_{B,n}|||_{T_0} \leqslant h$ for all $n \in \mathbf{N}$. For any $\varepsilon > 0$, $x \in H$ there exist $N_{\varepsilon,x}$, $m_\varepsilon \in \mathbf{N}$ such that $q^m < \varepsilon$ for all $m \geqslant m_\varepsilon$ and $\|H_{n,m_\varepsilon}(t,s)x - G_{m_\varepsilon}(t,s)x\| \leqslant \varepsilon$ for any $n \geqslant N_{\varepsilon,x}$. Hence, for any $x \in H$ and $n \geqslant N_{\varepsilon,x}$, we get

$$\bigl\| U_{B,n}(t,s)x - U_B(t,s)x \bigr\| \leqslant \varepsilon\bigl( 2\bigl|\bigl|\bigl| U(t,s) \bigr|\bigr|\bigr|_{T_0} + |||U_B|||_{T_0} + h \bigr)\|x\| + \varepsilon.$$

That is for any $x \in H$, $U_{n,B}(t,s)x \to_{n\to\infty} U_B(t,s)x$ uniformly on $\Delta(T_0)$. The proof of the first step is finished.

*Step* II. The proof of the statement (2).

For any $T_1, T_2 \in (0,\infty)$, $T_1 \leqslant T_2$ we will use the notation $\Delta(T_1,T_2)$ for the set $\{(t,s),\ T_1 \leqslant s \leqslant t \leqslant T_2\}$. If $T_0$ is the number given by Step I, it is clear that there exists $k_T \in \mathbf{N}$ such that $T = k_T T_0 + r$, $0 \leqslant r < T_0$. Reasoning as above we can prove that the statement (2) holds on each interval $\Delta(T_0, 2T_0), \ldots, \Delta(kT_0, T)$.

Let $(t,s) \in \Delta(T)$. There exist $k_s, k_t \in \mathbf{N}$, $k_s < k_t \leqslant k_T$ such that $t \in [k_t T_0, (k_t+1)T_0]$, $s \in [(k_s-1)T_0, k_s T_0]$. By semigroup property we get

$$U_{B,n}(t,s)x = U_{B,n}(t,k_t T_0)\cdots U_{B,n}(k_s T_0, s)x \quad \text{and} \quad U_B(t,s)x = U_B(t,k_t T_0)\cdots U_B(k_s T_0, s)x.$$

Arguing as in the proof of the first step, it is easy to see that, as $n \to \infty$ the operators $U_{B,n}(t,k_t T_0), \ldots, U_{B,n}(k_s T_0, s)$ converge uniformly with respect to $(t, k_t T_0, \ldots, k_s T_0, s) \in \mathbf{R}^{k_t - k_s + 3}$ to the operators $U_B(t, k_t T_0), \ldots, U_B(k_s T_0, s)$, respectively. Using Lemma 3 (Remark 4) we deduce that $U_{B,n}(t,s)x \to_{n\to\infty} U_B(t,s)x$ uniformly with respect to $(t,s) \in \Delta(T)$. The proof is complete. □

Let us introduce the hypothesis:

**H1:** $U, U_n \in C_s(\Delta(T), L(H))$, $n \in \mathbf{N}$, are evolution operators satisfying $\lim_{n\to\infty} U_n(t,s)x = U(t,s)x$, $x \in H$, uniformly for $(t,s) \in \Delta(T)$ and $A_n \in C_s([0,T], L(H))$, $n \in \mathbf{N}$, such that $U_n$ is the unique strong solution of the equation

$$\frac{\partial x_n(t)}{\partial t} = A_n(t)x_n(t), \quad t \in (s,T],\ x_n(s) = x,\ x \in H. \tag{6}$$

**Remark 6.** The hypothesis (P1) in the above theorem could be replaced with the assumption (H1) and the conclusions of the theorem remain true.

In fact the strong differentiability of $U_n$ is necessary only to prove the strong differentiability of $U_{B,n}$. Hence the proof is valid if we remove the differentiability conditions both from the hypothesis and the conclusion of Theorem 5. We obtain the following result.

**Proposition 7.** *Assume that $U, U_n \in C_s(\Delta(T), L(H))$ are evolution operators, $B \in C_s(\Delta(T), L(H))$ and $\lim_{n\to\infty} U_n(t,s)x = U(t,s)x$ uniformly for $(t,s) \in \Delta(T)$. There exists a unique solution $U_B \in C_s(\Delta(T), L(H))$ (respectively $U_{B,n} \in C_s(\Delta(T), L(H))$) of Eq.* (2) (*respectively* (3)). *Moreover, $U_B$ and $U_{B,n}$ are evolution operators and* $\lim_{n\to\infty} U_{B,n}(t,s)x = U_B(t,s)x$ *uniformly for $(t,s) \in \Delta(T)$.*

## 4. Evolution operators in $\mathcal{H}_2$

We recall that (P1), (P2) hold. For all $n \in \mathbf{N}$ and $t \geqslant 0$ we consider the mappings

$$\mathcal{A}_n(t) : \mathcal{H}_2 \to \mathcal{H}_2, \quad \mathcal{A}_n(t)(P) = A_n(t)P + PA_n^*(t), \quad P \in \mathcal{H}_2,$$

$$L(t) : \mathcal{H}_2 \to \mathcal{H}_2, \quad L(t)(P) = B(t)P + PB^*(t) + \sum_{i=1}^{m} G_i(t)PG_i^*(t), \quad P \in \mathcal{H}_2.$$

It is easy to verify that $\mathcal{A}_n(t), L(t) \in L(\mathcal{H}_2)$ for all $t \geqslant 0$. Let $\mathcal{A}_n^*(t), L^*(t) : \mathcal{H}_2 \to \mathcal{H}_2$,

$$\mathcal{A}_n^*(t)(P) = A_n^*(t)P + PA_n(t), \quad P \in \mathcal{H}_2,$$

$$L^*(t)(P) = B^*(t)P + PB(t) + \sum_{i=1}^{m} G_i^*(t)PG_i(t), \quad P \in \mathcal{H}_2,$$

be the adjoint operators of $\mathcal{A}_n(t)$ respectively $L(t)$ in $\mathcal{H}_2$ for all $t \in [0, T]$.

**Lemma 8.**

(a) $\mathcal{A}_n \in C([0,T], L(\mathcal{H}_2))$, $n \in \mathbf{N}$. *In addition, if* $\mathrm{P1}(A^*)$ *holds then* $\mathcal{A}_n^* \in C([0,T], L(\mathcal{H}_2))$, $n \in \mathbf{N}$.
(b) $L \in C_s([0,T], L(\mathcal{H}_2))$. *Moreover, if* (P3) *holds then* $L^* \in C_s([0,T], L(\mathcal{H}_2))$.

**Proof.** (a) Since $A_n \in C([0,T], L(H))$ (see [17, Lemma 3]) it is clear that $\mathcal{A}_n \in C([0,T], L(\mathcal{H}_2))$, $n \in \mathbf{N}$. The statement for $\mathcal{A}_n^*$ follows similarly.

(b) Let $P \in \mathcal{H}_2$. Using (P2), Lemma 1 and [17, Lemma 3], we deduce that the function $t \to L(t)(P)$, $t \in [0,T]$, belongs to $C([0,T], \mathcal{H}_2)$. It follows that $L \in C_s([0,\infty), L(\mathcal{H}_2))$, $n \in \mathbf{N}$. Analogously we can prove that if (P3) holds, $L^* \in C_s([0,T], L(\mathcal{H}_2))$. The proof is complete. □

The next result is a version of Lemma 1 in [4]. Unlike our case, Lemma 1 in [4] is proved for sequences in $C_s([0,T], L(H))$, but in both situations the proof is essentially the same and will be omitted.

**Lemma 9.** *Let $F_n \in C_s(\Delta(T), L(H))$, $n \in \mathbf{N}$, be a sequence of functions strongly convergent to $\{F(t,s),\ (t,s) \in \Delta(T)\}$, uniformly on $\Delta(T)$. If $P \in \mathcal{H}_2$, then $\{F_n(t,s)P\}_{n\in\mathbf{N}}$ is $\|.\|_2$, convergent to $F(t,s)P$, uniformly with respect to $(t,s) \in \Delta(T)$.*

**Theorem 10.**

(a) *For any $n \in \mathbf{N}$, there exists an evolution operator $V_n(t,s) \in C(\Delta(T), L(\mathcal{H}_2))$, which is the unique strong solution (on $\Delta(T)$) of the equation*

$$\frac{\partial X_n(t)}{\partial t} = \mathcal{A}_n(t)X_n(t), \quad t \in (s,T], \qquad X_n(s) = X, \quad X \in \mathcal{H}_2. \tag{7}$$

*Moreover,* $V_n(t,s)(X) = U_n(t,s)XU_n^*(t,s)$ *for any* $X \in \mathscr{H}_2$ *and*

$$V_n(t,s)(X) \underset{n\to\infty}{\rightarrow} V(t,s)(X) = U(t,s)XU^*(t,s) \tag{8}$$

*uniformly on* $\Delta(T)$. *Here the convergence is in* $\|.\|_2$.

(b) *There exists a unique strong solution* $V_{L,n} \in C_s(\Delta(T), L(\mathscr{H}_2))$ *of the equation*

$$\frac{\partial P_n(t)}{\partial t} = \mathscr{A}_n(t)P_n(t) + L(t)P_n(t), \quad t \in (s,T], \qquad P_n(s) = X \in \mathscr{H}_2,$$

*for any* $n \in \mathbf{N}$. $V_{L,n}$ *is the perturbed evolution operator corresponding to the perturbation* $L$ *and satisfies the following integral equation*

$$V_{L,n}(t,s)X = V_n(t,s)X + \int_s^t V_n(t,r)L(r)V_{L,n}(r,s)X\,dr, \quad X \in \mathscr{H}_2.$$

(c) *There exists an evolution operator* $V_L(t,s)$ *which is the unique solution in* $C_s(\Delta(T), L(\mathscr{H}_2))$ *of the equation*

$$V_L(t,s)X = V(t,s)X + \int_s^t V(t,r)L(r)V_L(r,s)X\,dr, \quad X \in \mathscr{H}_2. \tag{9}$$

(d) *For any* $X \in \mathscr{H}_2$, $V_{L,n}(t,s)(X) \rightarrow_{n\to\infty} V_L(t,s)(X)$, *uniformly on* $\Delta(T)$ *in* $\mathscr{H}_2$.

**Proof.** (a) The existence of the unique strong solution of (7) in $C(\Delta(T), L(\mathscr{H}_2))$, given by $V_n(t,s)(X) = U_n(t,s)XU_n^*(t,s)$ is a consequence of Lemma 8(a) and Theorem 5.2 in [7]. It is clear that $V_n(t,s)$ is an evolution operator for all $n \in \mathbf{N}$. We only have to prove the strong convergence of $V_n(t,s)$ to $V(t,s)$, as $n \to \infty$, uniformly with respect to $(t,s) \in \Delta(T)$. Let $X \in \mathscr{H}_2$. Using the properties of the norm $\|.\|_2$, we get

$$\begin{aligned}&\bigl\|U_n(t,s)XU_n^*(t,s) - U(t,s)XU^*(t,s)\bigr\|_2\\&\quad \leqslant \bigl\|U_n(t,s)\bigr\|\bigl\|XU_n^*(t,s) - XU^*(t,s)\bigr\|_2 + \bigl\|U(t,s)\bigr\|\bigl\|U_n(t,s)X - U(t,s)X\bigr\|_2.\end{aligned}$$

Now, we use Lemma 9 and (1) to deduce that for any $X \in \mathscr{H}_2$,

$$\bigl\|U_n(t,s)X - U(t,s)X\bigr\|_2 \underset{n\to\infty}{\rightarrow} 0$$

uniformly on $\Delta(T)$. Consequently $\|XU_n^*(t,s) - XU^*(t,s)\|_2 \rightarrow_{n\to\infty} 0$, uniformly on $\Delta(T)$. Since there exists $M > 0$ such that $\|U_n(t,s)\|, \|U(t,s)\| < M$ for all $(t,s) \in \Delta(T)$, $n \in \mathbf{N}$, we see that

$$\bigl\|U_n(t,s)XU_n^*(t,s) - U(t,s)XU^*(t,s)\bigr\|_2 \underset{n\to\infty}{\rightarrow} 0$$

uniformly on $\Delta(T)$. Hence (8) holds. The conclusion follows.

The statements (b), (c) and (d) are direct consequences of Proposition 7 and Remark 6. Indeed, using Lemma 1 we deduce that if $F \in C_s(\Delta(T), L(H))$ and $X \in \mathscr{H}_2$ then $FX \in C(\Delta(T), \mathscr{H}_2)$. Hence $V(t,s)(X) \in C(\Delta(T), \mathscr{H}_2)$, for all $X \in \mathscr{H}_2$ and consequently $V(t,s) \in C_s(\Delta(T), L(\mathscr{H}_2))$. Now it is easy to see that $V(t,s)$, $V_n(t,s)$ are evolution operators on the Hilbert space $\mathscr{H}_2$. Since $L \in C_s(\Delta(T), L(\mathscr{H}_2))$, (7) and (8) hold, we deduce that the hypotheses of Proposition 7 and Remark 6 are fulfilled and consequently (b), (c) and (d) hold. The proof is complete. □

The evolution operator $V_L(t,s)$ is the perturbed evolution operator corresponding to the perturbation $L$.

The following proposition shows that the conclusions of the above theorem are true under a more general hypothesis.

**Proposition 11.** *If* (H1), (P2) *hold then all the conclusions of the above theorem remain true excepting the assertion* $V_n(t,s) \in C(\Delta(T), L(\mathscr{H}_2))$ *in statement* (a) *of the theorem, which must be replaced by* $V_n(t,s) \in C_s(\Delta(T), L(\mathscr{H}_2))$.

**Proof.** If $A_n \in C_s([0,T], L(H))$, $n \in \mathbf{N}$, then reasoning as in the proof of Lemma 8 statement (b) it follows $\mathscr{A}_n \in C_s([0,T], L(\mathscr{H}_2))$, $n \in \mathbf{N}$. We deduce by Proposition 5 in [17] that there exists a unique strong solution of (7) in $C_s(\Delta(T), L(\mathscr{H}_2))$, given by $V_n(t,s)(X) = U_n(t,s)XU_n^*(t,s)$. Now arguing exactly as in the proof of the above theorem we obtain the conclusion. □

**Remark 12.**

(i) Assume that (P2) holds, $A$ is the infinitesimal generator of a $C_0$-semigroup in $H$ and $A_n$, $n \in \mathbf{N}$, are the Yosida approximations of $A$. If $A(t) = A$, $A_n(t) = A_n$, $t \in [0,\infty)$, then the evolution operators $U, U_n$, $n \in \mathbf{N}$, generated by the families $A$, $A_n$ are $C_0$-semigroups and by Theorem 1.5.5 in [7] it follows that (H1) holds. Then the conclusions of Theorem 10 stay true.

(ii) If $A \in C_s([0,T], L(H))$ and (P2) holds then the statement (c) of the above theorem is true and $V_L \in C_s(\Delta(T), L(\mathscr{H}_2))$ is the unique strong solution of the equation $\frac{\partial P(t)}{\partial t} = \mathscr{A}(t)P(t) + L(t)P(t)$, $t \in (s,T]$, $P(s) = X \in \mathscr{H}_2$.

## 5. Stochastic differential equations in Hilbert spaces

Let $(\Omega, F, \mathscr{F}_t, t \in [0,\infty), P)$ be a stochastic basis and let $L_t^2(H) = L^2(\Omega, \mathscr{F}_t, P, H)$, $t \in [0,\infty)$, be the space of all equivalence class of $H$-valued, $\mathscr{F}_t$-measurable random variables $\xi$, such that $E\|\xi\|^2 < \infty$. We consider the following stochastic equation (denoted $\{A, B, G_i\}$),

$$\begin{aligned} &dy(t) = A(t)y(t)\,dt + B(t)y(t)\,dt + \sum_{i=1}^{m} G_i(t)y(t)\,dw_i(t), \\ &y(s) = \xi \in L_s^2(H), \end{aligned} \tag{10}$$

where the coefficients $A(t)$ and $G_i(t)$, $i = 1,\dots,m$, $m \in \mathbf{N}^*$, satisfy the hypotheses (P1), (P2) and $w_i$, $i = 1,\dots,m$, are independent real Wiener processes relative to $\mathscr{F}_t$.

Let us consider $T > 0$. It is known (see [9,16]) that (10) has a unique mild solution in $C([s,T]; L_2(\Omega, H))$ that is adapted to $\mathscr{F}_t$, namely the solution of the integral equation

$$y(t) = U(t,s)\xi + \int_s^t U(t,r)B(r)y(r)\,dr + \int_s^t \sum_{i=1}^{m} G_i(r)y(r)\,dw_i(r), \quad \xi \in L_s^2(H).$$

We associate to (10) the approximating system:

$$\begin{aligned} &dy_n(t) = A_n(t)y_n(t)\,dt + B(t)y_n(t)\,dt + \sum_{i=1}^{m} G_i(t)y_n(t)\,dw_i(r), \\ &y_n(s) = \xi \in L_s^2(H), \end{aligned} \tag{11}$$

where $A_n(t)$, $n \in \mathbf{N}$ are the Yosida approximations of $A(t)$.

We will denote by $y(t,s;\xi)$ (respectively $y_n(t,s,\xi)$) the unique mild solution of (10) (respectively classical solution of (11)) with the initial condition $y(s) = \xi$ (respectively $y_n(s) = \xi$), $\xi \in L_s^2(H)$ (see [12]).

**Lemma 13.** *(See [12,16].) There exists a unique mild (respectively classical) solution to* (10) *(respectively* (11)*) and $y_n \to y$ in mean square uniformly on any bounded subset of* $[s,\infty)$.

### *5.1. Representation theorem for the covariance operator associated to the solution of the linear stochastic differential equation*

Let $\eta \in L^2(\Omega, F, P, H)$. We denote by $E(\eta \otimes \eta)$ the linear and bounded operator which act on $H$, given by $E(\eta \otimes \eta)(x) = E(\langle x, \eta\rangle \eta)$, $x \in H$.

The operator $E(\eta \otimes \eta)$ is called the covariance operator of the random variable $\eta$. It is known (see [17]) that

$$E\|\eta\|^2 = \big\|E(\eta \otimes \eta)\big\|_1.$$

The following result is known [17].

**Proposition 14.** *If $y_n(t,s,\xi)$, $\xi \in L_s^2(H)$ is the classical solution of* (11) *then $E[y_n(t,s,\xi) \otimes y_n(t,s,\xi)]$ is the unique classical solution of the following initial value problem*

$$\frac{dP_n(t)}{dt} = \mathscr{A}_n(t)P_n(t) + L(t)P_n(t), \quad t > s \geqslant 0, \tag{12}$$

$$P_n(s) = E(\xi \otimes \xi). \tag{13}$$

**Theorem 15.** *Assume that* (P1), (P2) *hold. If $y(t,s,\xi)$, $\xi \in L_s^2(H)$ is the mild solution of* (10) *and $V_L(t,s)$ is the perturbed evolution operator introduced by Theorem* 10, *then for all $(t,s) \in \Delta(T)$,*

$$E\big[y(t,s,\xi) \otimes y(t,s,\xi)\big] = V_L(t,s)E(\xi \otimes \xi). \tag{14}$$

**Proof.** It is clear that $E(\xi \otimes \xi) \in C_1(H) \subset C_2(H)$ and obviously $E(\xi \otimes \xi)$ is a self adjoint operator. Hence we apply Theorem 10 to deduce that the unique strong solution of (12), (13) is $V_{L,n}(t,s)E(\xi \otimes \xi)$ and $V_{L,n}(t,s)E(\xi \otimes \xi) \to_{n\to\infty} V_L(t,s)E(\xi \otimes \xi)$, uniformly on $\Delta(T)$ in $\|.\|_2$.

Using Proposition 14, we deduce that

$$E\big[y_n(t,s,\xi) \otimes y_n(t,s,\xi)\big] \underset{n\to\infty}{\to} V_L(t,s)E(\xi \otimes \xi)$$

uniformly on $\Delta(T)$ in $\|.\|_2$.

On the other hand, by Lemma 13 it follows (see also the proof of Theorem 8 in [17]) that for all $u \in H$, $\langle E[y_n(t,s,\xi) \otimes y_n(t,s,\xi)]u, u\rangle \to_{n\to\infty} \langle E[y(t,s,\xi) \otimes y(t,s,\xi)]u, u\rangle$ uniformly on any bounded subset of $[s,\infty)$. Since the sequence $E[y_n(t,s,\xi) \otimes y_n(t,s,\xi)]$ converges in $\|.\|_2$ to $V_L(t,s)E(\xi \otimes \xi)$, then it converges in the weak topology to the same limit and we deduce that $E[y(t,s,\xi) \otimes y(t,s,\xi)] = V_L(t,s)E(\xi \otimes \xi)$. The proof is complete. □

*The periodic case*

Let us introduce the following hypothesis:

**P4:** *There exists $\tau > 0$ such that $A(t) = A(t+\tau)$, $B(t) = B(t+\tau)$, $G_i(t) = G_i(t+\tau)$, $i = 1, \ldots, m$, for all $t \geqslant 0$.*

It is known (see [10,19]) that if (P1), (P2) and (P4) hold then the evolution operator generated by the family $A(t)$, $t \geqslant 0$, is $\tau$-periodic, that is

$$U(t+\tau, s+\tau) = U(t,s) \quad \text{for all } t \geqslant s \geqslant 0. \tag{15}$$

**Proposition 16.** *Assume that* (P1), (P2), (P4) *hold. Then*

(a) *the evolution operator $V_L(t,s)$ introduced in Theorem* 10 *is $\tau$-periodic*;
(b) *the covariance operator associated to the mild solution $y(t,s,\xi)$, $\xi \in L_s^2(H)$ of* (10) *is $\tau$-periodic and is given by* (14).

**Proof.** It is clear that (b) is a direct consequence of (a) and of Theorem 15. Hence we only prove (a). By the hypotheses we deduce that $V(t,s)$ is $\tau$-periodic.

Using (9) and changing the variable it follows that for all $X \in \mathscr{H}_2$

$$\begin{aligned} V_L(t+\tau, s+\tau)X &= V(t+\tau, s+\tau)X + \int_{s+\tau}^{t+\tau} V(t+\tau, r)L(r)V_L(r, s+\tau)X\,dr \\ &= V(t,s)X + \int_s^t V(t+\tau, u+\tau)L(u+\tau)V_L(u+\tau, s+\tau)X\,du. \end{aligned}$$

By (P4) we get

$$V_L(t+\tau,s+\tau)X = V(t,s)X + \int_s^t V(t,u)L(u)V_L(u+\tau,s+\tau)X\,du.$$

Since (9) has a unique solution in $C_s(\Delta(T), L(\mathscr{H}_2))$ we see that $V_L(t+\tau,s+\tau) = V_L(t,s)$ for all $t \geqslant s \geqslant 0$. The proof of the statement (a) is complete. □

Using Remark 12(i) we deduce that all the results of this section remain true if we replace hypothesis (P1) with the assumption that $A$ is the infinitesimal generator of a $C_0$-semigroup in $H$ and $A(t) = A$, $t \in [0,\infty)$. Also the conclusions of Theorem 15 and Proposition 16 remain true if we remove (P1) and we assume $A \in C_s([0,T], L(H))$ (see Remark 12(ii)).

### 5.2. *Characterizations of the stochastic uniform observability*

If $C \in C_s([0,\infty), L(H,V))$, we consider the system $\{A,B,G_i;C\}$ formed by equation $\{A,B,G_i\}$ and the observation relation $z(t) = C(t)y(t,s,x)$. In that follows we will assume that (P2), (P3) hold for $T = \infty$.

**Definition 17.** (See [13].) The system $\{A,B,G_i;C\}$ is stochastically uniformly observable if there exist $\sigma > 0$ and $\gamma > 0$ such that

$$E\int_t^{t+\sigma} \|C(r)y(r,t;x)\|^2\,dr \geqslant \gamma\|x\|^2 \tag{16}$$

for all $t \in \mathbf{R}$ and $x \in H$.

Assume that $\mathscr{C} \in C([0,\infty),\mathscr{H}_2)$, $\mathscr{C}(t) \geqslant 0$, $t \geqslant 0$ and let $T > 0$ be fixed. For all $t \geqslant 0$, we consider the function $\mathscr{L}(t) : L(H) \to L(H)$,

$$\mathscr{L}(t)(P) = B^*(t)P + PB(t) + \sum_{i=1}^m G_i^*(t)PG_i(t), \quad P \in L(H). \tag{17}$$

We introduce the following Lyapunov equation

$$\begin{aligned}&\frac{dX(t)}{dt} + A^*(t)X(t) + X(t)A(t) + \mathscr{L}(t)X(t) + \mathscr{C}(t) = 0,\\ &X(T) = 0 \in L(H).\end{aligned} \tag{18}$$

According with [12] (and Theorem 5), we say that $X$ is a mild solution on $[0,T]$ of (18), if $X \in C_s([0,T], L^+(H))$ and if for all $t \in [0,T]$ and $x \in H$ it satisfies

$$X(t)x = \int_t^T U_B^*(r,t)\left[\sum_{i=1}^m G_i^*(r)PG_i(r) + \mathscr{C}(r)\right]U_B(r,t)x\,dr,$$

where $U_B$ is the perturbed evolution operator corresponding to the perturbation $B$.

If $A_n(t)$, $n \in \mathbf{N}$ are the Yosida approximations of $A(t)$ then we introduce the approximating equation

$$\begin{aligned}&\frac{dX_n(t)}{dt} + A_n^*(t)X_n(t) + X_n(t)A_n(t) + \mathscr{L}(t)X_n(t) + \mathscr{C}(t) = 0,\\ &X_n(T) = 0 \in L(H).\end{aligned} \tag{19}$$

It is known (see [12, Lemma 3] and Theorem 5) that there exists a unique mild (respectively classical) solution $X$ (respectively $X_n$) of (18) (respectively (19)) on $[0,T]$ such that $X(T) = 0$ (respectively $X_n(T) = 0$) denoted $X(T,t;0)$ (respectively $X_n(T,t;0)$) and for each $x \in H$,

$$X_n(T,s,0)x \underset{n\to\infty}{\to} X(T,s,0)x \tag{20}$$

uniformly on $[0,T]$.

It is not difficult to see that $X_n$, the unique classical solution of (19) on $[0,T]$ is also the unique solution in $C_s([0,T],L^+(H))$ of the following integral equation

$$X_n(t)x=\int_t^T U_n^*(r,t)\big[\mathscr{L}(r)X_n(r)+\mathscr{C}(r)\big]U_n(r,t)x\,dr.$$

Using (20) and passing to the limit for $n\to\infty$ in the above relation we deduce that the mild solution on $[0,T]$ of (18) is the unique solution in $C_s([0,T],L^+(H))$ of

$$X(t)x=\int_t^T U^*(r,t)\big[\mathscr{L}(r)X(r)+\mathscr{C}(r)\big]U(r,t)x\,dr. \tag{21}$$

Therefore we could use Eq. (21) when we refer to the mild solution on $[0,T]$ of (18).

**Lemma 18.** *Assume that* (P1), P1($A^*$)*,* (P2), (P3) *hold and* $\mathscr{C}\in C([0,\infty),\mathscr{H}_2)$*,* $C(t)\geqslant 0$ *for all* $t\geqslant 0$. *Then the unique mild solution* $X(T,t;0)$ *of the Lyapunov equation* (18) *is given by*

$$\big\langle X(T,t,0)x,x\big\rangle=\int_t^T\big\langle\mathscr{C}(p),V_L(p,t)(x\otimes x)\big\rangle_2\,dp,\quad x\in H,$$

*where* $V_L(p,t)$ *is the perturbed evolution operator introduced by Theorem* 10.

**Proof.** First we note that Eq. (19) considered in $\mathscr{H}_2$ could be rewritten

$$\begin{aligned}&\frac{dX_n(t)}{dt}+\mathscr{A}_n^*(t)X_n(t)+L^*(t)X_n(t)+\mathscr{C}(t)=0,\quad t\in[0,T],\\&X_n(T)=0\in\mathscr{H}_2,\end{aligned} \tag{22}$$

where $\mathscr{A}_n^*(t)$ and $L^*(t)$ are the adjoint operators of $\mathscr{A}_n(t)$ respectively $L(t)$ in $\mathscr{H}_2$.

Changing the variable in (22) $r=T-t$ and denoting $Y_n(r)=X_n(T-r)$ we obtain the following equation:

$$\begin{aligned}&\frac{dY_n(r)}{dr}=\big[\mathscr{A}_n^*(T-r)+L^*(T-r)\big]Y_n(r)+\mathscr{C}(T-r),\quad r\in[0,T],\\&Y_n(0)=0\in\mathscr{H}_2.\end{aligned}$$

By Lemma 8 in this paper, [17, Proposition 5] and [1, Theorem 9.9], it follows that $r\to[\mathscr{A}_n^*(T-r)+L^*(T-r)]$ is strongly continuous on $\mathscr{H}_2$ and generates the evolution operator $\mathscr{V}(r,s)=V_{L,n}^*(T-s,T-r)$, $(r,s)\in\Delta(T)$. Here $V_{L,n}^*(t,s)$ is the adjoint operator in $\mathscr{H}_2$ of the evolution operator $V_{L,n}(r,t)$ introduced in Theorem 10. Consequently, the above equation has a unique classical solution [7]

$$Y_n(r)=\int_0^r V_{L,n}^*(T-\sigma,T-r)\mathscr{C}(T-\sigma)\,d\sigma.$$

By a simple computation we get

$$\begin{aligned}\big\langle Y_n(r),x\otimes x\big\rangle_2&=\int_0^r\big\langle V_{L,n}^*(T-\sigma,T-r)\mathscr{C}(T-\sigma),x\otimes x\big\rangle_2\,d\sigma\\&=\int_0^r\big\langle\mathscr{C}(T-\sigma),V_{L,n}(T-\sigma,T-r)(x\otimes x)\big\rangle_2\,d\sigma,\\\big\langle Y_n(r)x,x\big\rangle&=\int_{T-r}^T\big\langle\mathscr{C}(p),V_{L,n}(p,T-r)(x\otimes x)\big\rangle_2\,dp.\end{aligned}$$

Hence

$$\langle X_n(t)x, x\rangle = \int_t^T \langle \mathscr{C}(p), V_{L,n}(p,t)(x\otimes x)\rangle_2\, dp \geqslant 0.$$

It is clear that the classical solution of (22) considered in $\mathscr{H}_2$ is also a classical solution in $L^+(H)$ and coincides with the classical solution $X_n(T,t;0)$ of (19) given by Lemma 3 in [12]. As $n\to\infty$ in the above relation and taking into account Theorem 10(d) and (20) we get

$$\langle X(T,t,0)x, x\rangle = \int_t^T \langle \mathscr{C}(p), V_L(p,t)(x\otimes x)\rangle_2\, dp, \tag{23}$$

where $V_L$ is the perturbed evolution operator defined by (10). The proof is complete. □

The following theorem is the main result of this section. It gives a deterministic characterization of the stochastic uniform observability.

**Theorem 19.** *Assume that $C\in C_s([0,\infty), L(H,V))$, $C^*\in C_s([0,\infty), L(V,H))$ and* (P1), P1($A^*$), (P2), (P3) *hold. The following statements are equivalent*:

(1) *$\{A, B, G_i; C\}$ is stochastically uniformly observable*;
(2) *there exist $\sigma>0$, $\gamma>0$ such that $X(T-\sigma)\geqslant \gamma I$, for all $T\geqslant\sigma$, where $X\in C_s([0,T], L^+(H))$ is the unique mild solution of the problem*

$$\frac{dX(t)}{dt} + A^*(t)X(t) + X(t)A(t) + \mathscr{L}(t)X(t) + C^*(t)C(t) = 0, \tag{24}$$

$$X(T) = 0 \in L(H). \tag{25}$$

(*We recall that the function $\mathscr{L}$ is defined by* (17).)

**Proof.** Let $\{e_1, e_2, \dots, e_n, \dots\}$ be an orthonormal basis of $V$. We consider the sequence of operators $\mathscr{C}_n(t) = \sum_{k=1}^n C^*(t)(e_k)\otimes C^*(t)(e_k)$, $n\in\mathbf{N}$. It is easy to see that $\mathscr{C}_n(t)\in C([0,\infty),\mathscr{H}_2)$. Indeed for all $t, t_0\in[0,\infty)$ we have

$$\begin{aligned}\|\mathscr{C}_n(t)-\mathscr{C}_n(t_0)\|_2 &\leqslant \sum_{k=1}^n \|C^*(t)(e_k)\otimes[C^*(t)-C^*(t_0)](e_k)\|_2 + \|[C^*(t)-C^*(t_0)](e_k)\otimes C^*(t_0)(e_k)\|_2\\ &\leqslant \sum_{k=1}^n \|C^*(t)(e_k)\|\,\|[C^*(t)-C^*(t_0)](e_k)\| + \|C^*(t_0)(e_k)\|\,\|[C^*(t)-C^*(t_0)](e_k)\|.\end{aligned}$$

Since $C^*\in C_s([0,\infty), L(V,H))$ it follows the conclusion.

Now we introduce the following system of differential equations on $L(H)$

$$\frac{dR_n(t)}{dt} + A^*(t)R_n(t) + R_n(t)A(t) + \mathscr{L}(t)R_n(t) + \mathscr{C}_n(t) = 0, \tag{26}$$

$$R_n(T) = 0\in L(H),\quad n\in\mathbf{N},\ t\in[0,T]. \tag{27}$$

We apply Lemma 18 to deduce that the unique mild solution of (26), (27) is

$$\langle R_n(t)x, x\rangle = \int_t^T \langle \mathscr{C}_n(p), V_L(p,t)(x\otimes x)\rangle_2\, dp \tag{28}$$

$$= \int_t^T \mathrm{Tr}\big[\mathscr{C}_n(p)V_L(p,t)(x\otimes x)\big]\, dp. \tag{29}$$

We see that $0 \leqslant \mathscr{C}_1(t) \leqslant \cdots \leqslant \mathscr{C}_n(t) \leqslant \mathscr{C}_{n+1}(t) \leqslant \cdots \leqslant \sup_{t\in[0,T]} \|C(t)\|^2 I_H$ for all $t \in [0,T]$. It is easy to verify (see [18, Lemma 1]) that the sequence $\mathscr{C}_n(t)$ converges strongly to $C^*(t)C(t)$, uniformly with respect to $t \in [0,T]$. On the other hand (14) ensure us that $V_L(p,t)(x \otimes x) \in C_1(H)$. Since for all $S \in C_1(H)$, $|\operatorname{Tr} S| \leqslant \|S\|_1$ we get

$$\begin{aligned}\left|\operatorname{Tr} C^*(p)C(p)V_L(p,t)(x\otimes x) - \operatorname{Tr}\mathscr{C}_n(p)V_L(p,t)(x\otimes x)\right| &= \left|\operatorname{Tr}\left[\left(C^*(p)C(p) - \mathscr{C}_n(p)\right)V_L(p,t)(x\otimes x)\right]\right| \\ &\leqslant \left\|\left[C^*(p)C(p) - \mathscr{C}_n(p)\right]V_L(p,t)(x\otimes x)\right\|_1.\end{aligned}$$

By Lemma 1 it follows that for all $x \in H$, $\|[C^*(p)C(p) - \mathscr{C}_n(p)]V_L(p,t)(x\otimes x)\|_1$ converges to 0 as $n \to \infty$, uniformly with respect to $(p,t) \in \Delta(T)$. Consequently, for all $x \in H$,

$$\operatorname{Tr}\left[\mathscr{C}_n(p)V_L(p,t)(x\otimes x)\right] \underset{n\to\infty}{\to} \operatorname{Tr}\left[C^*(p)C(p)V_L(p,t)(x\otimes x)\right]$$

uniformly with respect to $(p,t) \in \Delta(T)$. Therefore the right member of (29) converges to

$$\int_t^T \operatorname{Tr}\left[C^*(p)C(p)V_L(p,t)(x\otimes x)\right] dp \geqslant 0.$$

Using again the monotonicity of the sequence $\{\mathscr{C}_n(t)\}_{n\in\mathbf{N}^*}$ and (28) we deduce that $\{R_n(t)\}_{n\in\mathbf{N}^*}$ is monotone increasing, bounded above and strongly convergent to $R \in C_s([0,T], L^+(H))$, uniformly with respect to $t \in [0,T]$.

We will prove that $R$ coincides with $X$, the mild solution of (24), (25).

Indeed, if $X(t) = X(T,t;0)$ is the solution of (24), (25), then for any $x \in H$, $X(t)x - R_n(t)x \to_{n\to\infty} X(t)x - R(t)x$ uniformly with respect to $t \in [0,T]$.

On the other hand the difference $Z_n(t) = X(t) - R_n(t)$ is the solution of the following Lyapunov equation

$$\begin{aligned}&\frac{dZ_n(t)}{dt} + A^*(t)Z_n(t) + Z_n(t)A(t) + \mathscr{L}(t)Z_n(t) + C^*(t)C(t) - \mathscr{C}_n(t) = 0,\\ &Z_n(T) = 0.\end{aligned}$$

Since $C^*(t)C(t) - \mathscr{C}_n(t) \in C_s([0,\infty), L^+(H))$ we apply Lemma 3 in [12] to deduce that $Z_n(t)$ is the solution of the following integral equation

$$Z_n(t)x = \int_t^T U^*(s,t)\left[\mathscr{L}(s)Z_n(s) + C^*(s)C(s) - \mathscr{C}_n(s)\right]U^*(s,t)x\, dr.$$

Letting $n \to \infty$ in the above formula we get

$$X(t)x - R(t)x = \int_t^T U^*(s,t)\mathscr{L}(s)\left(X(s) - R(s)\right)U^*(s,t)x\, dr.$$

By Gronwall's inequality it follows that

$$X(t) - R(t) = 0$$

for all $t \in [0,T]$ and we obtain the conclusion.

As $n \to \infty$ in (29) we get

$$\left\langle X(T,t;0)x, x\right\rangle = \int_t^T \operatorname{Tr} C^*(p)C(p)V_L(p,t)(x\otimes x)\, dp. \tag{30}$$

We note that by Theorem 15 it follows that for all $\sigma > 0$, $t \geqslant 0$ and $x \in H$ we have

$$
\begin{aligned}
E \int_t^{t+\sigma} \left\| C(p) y(p,t;x) \right\|^2 dp
&= \int_t^{t+\sigma} \left\| C(p) E\big[y(p,t;x) \otimes y(p,t;x)\big] C^*(p) \right\|_1 dp = \int_t^{t+\sigma} \left\| C(p) V_L(p,t)(x \otimes x) C^*(p) \right\|_1 dp \\
&= \int_t^{t+\sigma} \operatorname{Tr} C^*(p) C(p) V_L(p,t)(x \otimes x)\, dp.
\end{aligned}
$$

Using (30) we see that

$$
E \int_t^{t+\sigma} \left\| C(p) y(p,t;x) \right\|^2 dr = \big\langle X(t+\sigma, t; 0)x, x \big\rangle.
$$

Now it is clear that $\{A, B, G_i; C\}$ is stochastically uniformly observable iff there exist $\sigma > 0$ and $\gamma > 0$ such that $\langle X(T, T-\sigma, 0)x, x\rangle \geqslant \gamma \|x\|^2$ for all $x \in H$ and $T \geqslant \sigma$. The proof is complete. □

**Corollary 20.** *If $A$ is the infinitesimal generator of a $C_0$-semigroup in $H$, $A(t) = A$, $t \in [0,\infty)$, $C \in C_s([0,\infty), L(H,V))$, $C^* \in C_s([0,\infty), L(V,H))$ and* (P2), (P3) *hold then the conclusions of the above theorem stay true.*

**Proof.** We first recall that if $S(t)$, $t \geqslant 0$ is the $C_0$-semigroup generated by $A$ then $S^*(t)$ is also a $C_0$-semigroup, which infinitesimal generator is $A^*$ [1,7]. Moreover, if $S_n(t)$ is the $C_0$-semigroup generated by $A_n$, the Yosida approximation of $A$, then for any $x \in H$, $\lim_{n\to\infty} S_n^*(t)x = S^*(t)x$ uniformly on bounded subsets of $[0,\infty)$ [7]. Reasoning exactly as in the proof of the above theorem and using Proposition 11 it follows the conclusion. □

**Remark 21.** By Remark 12 it follows that the conclusions of the above theorem are true if we replace the hypotheses $\mathrm{P1}(A)$,$\mathrm{P1}(A^*)$ with the assumption $A, A^* \in C_s([0,\infty), L(H))$. (In this case we will not use approximating systems.)

If either $A$ is the infinitesimal generator of a $C_0$-semigroup in $H$ or $\mathrm{P1}(A)$, $\mathrm{P1}(A^*)$ hold and $A(t) = A$, $B(t) = B \in L(H)$, $G_i(t) = G_i \in L(H)$, $i = 1, \ldots, m$, $C(t) = C \in L(H,V)$ for all $t \geqslant 0$, we shall say that the stochastic observed system $\{A, B, G_i; C\}$ is in the time invariant case. The next result give a characterization of the stochastic observability of the system $\{A, B, G_i; C\}$ in the time invariant case. We note that particularizing the next result to the case of finite dimensional Hilbert spaces we recover Theorem 4.3 (statements (i)–(iii)) obtained in [2] (of course for the case of Markov processes with the state space $\mathscr{D} = \{1\}$). Hence our main result (Theorem 19) extends the results obtained in [2] to the case of infinite dimensional spaces.

**Corollary 22.** *Assume that the stochastic system $\{A, B, G_i; C\}$ is in the time invariant case. The following statements are equivalent*:

(a) *$\{A, B, G_i; C\}$ is stochastically uniformly observable*;
(b) *there exists $\sigma > 0$ such that $Z(\sigma) > 0$, where $Z \in C_s([0,\infty), L^+(H))$ is the mild solution of the problem*

$$
\frac{dZ(t)}{dt} = A^* Z(t) + Z(t) A + \mathscr{L} Z(t) + C^* C, \tag{31}
$$
$$
Z(0) = 0, \tag{32}
$$

*and $\mathscr{L}$ is given by* (17).

**Proof.** Theorem 19, respectively Corollary 20, show that (a) is equivalent with the following statement:

There exist $\sigma > 0$, $\gamma > 0$ such that $X(T-\sigma) \geqslant \gamma I$ for all $T \geqslant \sigma$, where $X = X(T, .; 0) \in C_s([0,T], L^+(H))$ is the unique mild solution of the problem

$$\frac{dX(t)}{dt} + A^* X(t) + X(t)A + \mathscr{L}X(t) + C^*C = 0, \tag{33}$$

$$X(T) = 0. \tag{34}$$

By (30) it follows

$$\langle X(T, T-\sigma; 0)x, x\rangle = \int_{T-\sigma}^{T} \operatorname{Tr} C^*C V_L(p, T-\sigma)(x \otimes x)\,dp = \int_0^{\sigma} \operatorname{Tr} C^*C V_L(u+T-\sigma, T-\sigma)(x \otimes x)\,du.$$

We note that (P4) holds in the time invariant case and Proposition 16 ensure us that the evolution operator $V_L$ is $\tau$-periodic for any $\tau \geqslant 0$. Hence $V_L(u+T-\sigma, T-\sigma) = V_L(u, 0)$ for all $T \geqslant \sigma$ and

$$\langle X(T, T-\sigma; 0)x, x\rangle = \int_0^{\sigma} \operatorname{Tr} C^*C V_L(u, 0)(x \otimes x)\,du = \langle X(\sigma, 0; 0)x, x\rangle.$$

Now it is clear that (a) holds iff there exists $\sigma > 0$ such that $X(\sigma, 0; 0) > 0$, where $X \in C_s([0,T], L^+(H))$ is the mild solution of the Lyapunov equation (33) with the final condition

$$X(\sigma) = 0. \tag{35}$$

If $A_n$, $n \in \mathbf{N}$, are the Yosida approximations of $A$ we consider the following approximating systems associated to (33), (34), respectively (31), (32)

$$\frac{dX_n(t)}{dt} + A_n^* X_n(t) + X_n(t)A_n + \mathscr{L}X_n(t) + C^*C = 0, \quad X_n(T) = 0, \tag{36}$$

$$\frac{dZ_n(t)}{dt} = A_n^* Z_n(t) + Z_n(t)A_n + \mathscr{L}Z_n(t) + C^*C, \quad Z_n(0) = 0. \tag{37}$$

Changing the variable $t = T - r$ in (36) and denoting $Y_n(r) = X_n(T-r)$ we see that $Y_n(r)$ is the unique solution of (37) on $[0,T]$. Hence $Z_n(r) = X_n(T-r)$ for all $r \in [0,T]$. By Theorem 4.1 in [8] and Lemma 3 in [12] it follows that for all $x \in H$, $Z_n(t)x \to_{n\to\infty} Z(t)x$, $X_n(t)x \to_{n\to\infty} X(T,t;0)x$, uniformly with respect to $t \in [0,T]$. Here $Z(t)$ is the unique mild solution of (31), (32).

Now it is clear that $Z(r) = X(T, T-r; 0)$ for all $r \in [0,T]$. Hence

$$X(\sigma, 0, 0) = X(\sigma, \sigma - \sigma; 0) = Z(\sigma)$$

and (a) holds iff there exists $\sigma > 0$ such that $Z(\sigma) = X(\sigma, 0; 0) > 0$ where $Z$ is the unique mild solution of (31), (32). The proof is complete. □

According to Remark 21 it follows that the conclusion of the above corollary stay true if we assume that $A(t) = A \in L(H)$, $t \geqslant 0$.

The following examples will illustrate the theory.

**Example 23.** We consider the following stochastic wave equation with Dirichlet boundary conditions

$$\begin{aligned} &dv_t(\xi, t) = \frac{\partial^2 v(\xi, t)}{\partial \xi^2}\,dt + \cos(t)v(\xi, t)\,dw(t), \quad t \geqslant 0,\ \xi \in [0,1], \\ &v(0,t) = 0, \quad v(1,t) = 0, \quad v(\xi, 0) = x_1(\xi), \quad v_t(\xi, 0) = x_2(\xi), \end{aligned} \tag{38}$$

where $w$ is a real Wiener process, together the observation relation

$$z(t) = v_t(\xi, t). \tag{39}$$

We will prove that (38)–(39) is stochastically uniformly observable.

The linear operator $\Lambda = -\frac{\partial^2}{\partial \xi^2}$, $D(\Lambda) = H_0^1(0,1) \cap H^2(0,1)$ is the infinitesimal generator of an analytic semigroup on the Hilbert space $L^2(0,1)$ [7]. We introduce the Hilbert space $H = D(\Lambda^{1/2}) \oplus L^2(0,1)$ endowed with the inner product

$$\langle x, u\rangle_H = \langle \Lambda^{1/2}x_1, \Lambda^{1/2}u_1\rangle_{L^2(0,1)} + \langle x_2, u_2\rangle_{L^2(0,1)},$$
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H.$$

Let us consider the linear operator on $H$,

$$A : D(\Lambda) \oplus D(\Lambda^{1/2}) \to H, \qquad A(x) = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

It is known (see [1,11]) that $A$ generates a contraction semigroup in $H$

$$S(t)(x) = \begin{pmatrix} \sum_{n=1}^{\infty} 2[\langle x_1, \phi_n\rangle_{L^2(0,1)} \cos n\pi t + \frac{1}{n\pi}\langle x_2, \phi_n\rangle_{L^2(0,1)} \sin n\pi t]\phi_n \\ \sum_{n=1}^{\infty} 2[-n\pi\langle x_1, \phi_n\rangle_{L^2(0,1)} \sin n\pi t + \langle x_2, \phi_n\rangle_{L^2(0,1)} \cos n\pi t]\phi_n \end{pmatrix},$$

where $\phi_n = \sin n\pi\xi$. We see that the system (38)–(39) can be written

$$dy(t) = Ay(t)\,dt + B(t)y(t)\,dt + G(t)y(t)\,dw(t),$$
$$z = C(t)y(t),$$

where $y = (y_1, y_2)^T$, $B = 0$, $G(t)(y) = (0, \cos(t)y_1)^T$, $C(t)y = y_2$, $C \in L(H, L^2(0,1))$. Obviously hypotheses (P2), (P3) hold and $A$ is the infinitesimal generator of a $C_0$-semigroup. By Corollary 20 it follows that the above system is stochastically uniformly observable iff *there exist* $\sigma > 0, \gamma > 0$ such that $X(T-\sigma) \geqslant \gamma I$, for all $T \geqslant \sigma$, where $X \in C_s([0,T], L^+(H))$ is the solution of the following integral equation (see (21))

$$X(T,t)x = \int_t^T S^*(r-t)\big[G^*(r)X(r)G(r) + C^*(r)C(r)\big]S(r-t)x\,dr.$$

Clearly

$$\begin{aligned}\langle X(T, T-\sigma)x, x\rangle &\geqslant \int_{T-\sigma}^{T} \|C(r)S(r-T+\sigma)x\|_{L^2(0,1)}^2\,dr \\ &= \sum_{n=1}^{\infty} n^2\pi^2\langle x_1,\phi_n\rangle^2\left[\sigma - \frac{\sin 2n\pi\sigma}{2n\pi}\right] - \langle x_1,\phi_n\rangle\langle x_2,\phi_n\rangle(1-\cos 2n\pi\sigma) \\ &\quad + \langle x_2,\phi_n\rangle^2\left[\sigma + \frac{\sin 2n\pi\sigma}{2n\pi}\right].\end{aligned}$$

Since $\|x\|_H^2$ is equivalent with $2\sum_{n=1}^{\infty} n^2\pi^2\langle x_1,\phi_n\rangle^2 + \langle x_2,\phi_n\rangle^2$ we deduce that there exist $\sigma > 0$, $\gamma > 0$ such that $X(T-\sigma) \geqslant \gamma I$, for all $T \geqslant \sigma$ if

$$4\left[\sigma - 2\gamma - \frac{\sin 2n\pi\sigma}{2n\pi}\right]\left[\sigma - 2\gamma + \frac{\sin 2n\pi\sigma}{2n\pi}\right]n^2\pi^2 \geqslant (1-\cos 2n\pi\sigma)^2 \quad \text{and} \quad \sigma - 2\gamma > \left|\frac{\sin 2n\pi\sigma}{2n\pi}\right|$$

for all $n \in \mathbf{N}$. Obviously if $\sigma = 3$, $\gamma = 1/2$ the above inequalities are satisfied. Applying Corollary 20 it follows that the system (38) is stochastically uniformly observable.

**Example 24.** Consider the parabolic equation

$$dy(\xi,t) = \frac{\partial^2 y(\xi,t)}{\partial \xi^2}\,dt - (1/2)t^2 y(\xi,t)\,dt + ty(\xi,t)\,dw(t), \quad t \geqslant 0,\ \xi \in [0,1],$$
$$y(0,t) = y(1,t) = 0, \tag{40}$$

where $w(t)$ is a real Wiener process, together the observation relation

$$z(\xi,t) = y(\xi,t). \tag{41}$$

For this example we take $H = V = L^2(0,1)$, $A = \frac{\partial^2}{\partial \xi^2}$, $D(A) = H_0^1(0,1) \cap H^2(0,1)$, $B(t) = -(1/2)t^2 I$, $G(t) = tI$ and $C(t) = I$. Since the operator $A$ is self adjoint and generates an analytic semigroup $S(t)$ on $H$ [7] it follows that P1$(A)$ and P1$(A^*)$ hold. It is easy to see that the eigenvalues of $A$ are $\lambda_n = -n^2\pi^2$, the corresponding eigenvectors are $\phi_n = \sqrt{2}\sin(\pi n\xi)$, $n \in \mathbf{N}^*$, and

$$S(t)x = \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t} \sin(\pi n\xi) \int_0^1 x(r)\sin(\pi n r)\,dr.$$

Clearly the hypotheses of Theorem 19 are satisfied and (24) is written

$$\frac{dX(t)}{dt} + AX(t) + X(t)A + I = 0, \quad X(T) = 0.$$

The mild solution $X(T, T-\sigma)$ of the above equation is given by

$$\langle X(T,T-\sigma)x, x\rangle = \int_0^\sigma \|S(u)x\|^2\,du = \sum_{n=1}^{\infty} 2\frac{1-e^{-2n^2\pi^2\sigma}}{2n^2\pi^2}\left(\int_0^1 x(r)\sin(\pi n r)\,dr\right)^2.$$

Since $\langle X(T,T-\sigma)\phi_n, \phi_n\rangle = \frac{1-e^{-2n^2\pi^2\sigma}}{2n^2\pi^2}$ we deduce that $\langle X(T,T-\sigma)\phi_n,\phi_n\rangle \to 0$ as $n \to \infty$ and there do not exist $\sigma > 0$, $\gamma > 0$ such that $X(T,T-\sigma) \geqslant \gamma I$ for all $T \geqslant \sigma$. Hence (40), (41) is not stochastically uniformly observable.

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